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Dynamical replica theoretic analysis of CDMA detection dynamics

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Abstract
We investigate the detection dynamics of the Gibbs sampler for code-division multiple access (CDMA) multiuser detection. Our approach is based upon dynamical replica theory which allows an analytic approximation to the dynamics. We use this tool to investigate the basins of attraction when phase coexistence occurs and conclude that good decoding past the spinodal point is not practically possible with this algorithm. We examine the efficacy of our method by doing a comparison with Monte Carlo simulations.

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1. Introduction

Mobile phone communication is now a key technology across large swathes of the world. One of the essential technological ingredients for its success is the ability for multiple users to share a single channel (i.e. many people can use the same channel to communicate between their mobile phones and a particular base station). Code-division multiple access (CDMA) [1–3] is a protocol that allows this multiple access to a single channel through each user modulating their signal (via the so-called spreading codes) before transmitting to the base station. The base station receives a mixture of these modulated signals, combined with additional channel noise, and the task is then to use knowledge of the spreading codes and received signal to reconstruct the original information. CDMA has been the subject of several studies in the past few years that have utilized the relationship between a model of the communication process and the statistical mechanics of fully connected disordered Ising spin systems to examine the posterior distribution of the original signal using Bayesian inference [4–7]. This gives access to maximum a posteriori (MAP) and maximum posterior marginal (MPM) decoding. Progress has also been made in terms of algorithms for decoding using message passing procedures

[8–10]. Recently, both density evolution and generating functional analysis have been used to analyse the *dynamics* of some detection algorithms for the parallel inference canceller [11, 12]. The first of these techniques makes the relatively strong approximation of a Gaussian local field, and ignores the Onsager reaction term in the local field; however, it is known to generally give relatively good results when the detection dynamics converge [11]. In contrast, the generating functional approach [12, 13] is *exact*, but the complexity, both analytically and numerically, increases rapidly with the number of time steps considered and thus it is only practically useful for examining the first few steps of the dynamics. In this paper we exploit the alternative approach of dynamical replica theory [14–16]. This allows us to treat the dynamics of a sequential update detection algorithm (namely the Gibbs sampler) working in continuous time, with an analytic approximation scheme that we expect to be superior to density evolution, and with a numerical effort that increases only linearly in time. It was noted in [4] that there exists a spinodal for both MPM and MAP decoding, prior to this the decoding problem has two locally stable solutions, one with good performance and one with relatively poor performance. This coexistence has practical implications since local search algorithms starting from an initial state with a relatively high error rate are closer to the poor solution (at least in terms of the error rate). To go beyond this qualitative argument, however, one really requires dynamical tools since the basins of attraction are dynamically defined concepts. The theory which we develop in this paper allows us to examine these concepts. We examine the theory for the Gibbs sampler as a prototype local search algorithm, not because we believe that it is necessarily the optimal algorithm for detection and decoding in CDMA-type problems. The dynamical replica approach we describe here is an approximation, but its justification is given by comparison with Monte Carlo simulations. Finally we note that the theory we develop here is also applicable with minor modifications to the linear Ising perceptron [10].

2. Model definitions and order parameter evolution equations

We consider the demodulation problem for the N-user direct-sequence binary phase-shift-keying (DS/BPSK) CDMA system, with the simplifying assumptions that the channel noise is additive white Gaussian, chip and symbol timing are perfectly synchronized across users and the output power of the users is perfectly equalized by power control. For details of the equilibrium statistical mechanical analysis of this model, see [4–6] .

We consider N users sending information bits $\sigma_i^0 \in \{-1, 1\}$ $\forall i = 1, \ldots, N$. Each user i has a binary spreading code $\{\eta_i^t : t = 1, \ldots, p\}$, $\eta_i^t \in \{-1, 1\}$ so that in symbol interval t, user i transmits $\eta_i^t \sigma_i^0$. We model the spreading code sequences to be independent quenched random variables with $\mathrm{Prob}[\eta_i^t = \pm 1] = \frac{1}{2}$ and take the zero mean additive white Gaussian noise $\{\nu^t : t = 1, \ldots, p\}$ to have variance N/β_s. Thus, at each chip time step $t \in \{1, \ldots, p\}$ the received signal at the base station y^t is given by

$$y^t = \sum_{i=1}^{N} \eta_i^t \sigma_i^0 + \nu^t. \tag{1}$$

Bayesian inference shows that posterior distribution of the original signal, given the noisy signal, is given by a Gibbs–Boltzmann distribution with temperature β and Hamiltonian

$$H(\boldsymbol{\sigma}) = \frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j - \sum_{i=1}^{N} f_i\sigma_i^0 \qquad J_{ij} = \frac{1}{N}\sum_t \eta_i^t\eta_j^t \qquad f_i = \frac{1}{N}\sum_{t=1}^{p}\eta_i^t y^t \tag{2}$$

where the signal $\{y^t\}$ and the spreading codes $\{\eta_i^t\}$ constitute quenched disorder. The temperature β is a free control parameter, MAP decoding corresponds to the limit $\beta \to \infty$

while MPM decoding corresponds to $\beta = \beta_s$ (the Nishimori temperature [6]), although in general β_s may not be known.

We examine the detection problem by using a spin system with Glauber dynamics to model the posterior distribution. In order to analytically study the dynamical evolution of this distribution analytically, we use the techniques of dynamical replica theory [14–16], at the level of a three-parameter approximation. With Glauber dynamics the time evolution of the microscopic state probability distribution $p_t(\boldsymbol{\sigma})$ is given by the master equation

$$\frac{\mathrm{d}}{\mathrm{d}t} p_t(\boldsymbol{\sigma}) = \sum_{k=1}^{N} [p_t(F_k\boldsymbol{\sigma}) w_k(F_k\boldsymbol{\sigma}) - p_t(\boldsymbol{\sigma}) w_k(\boldsymbol{\sigma})] \tag{3}$$

with the spin-flip operator $F_k\Phi(\boldsymbol{\sigma}) \equiv \Phi(\sigma_1, \ldots, -\sigma_k, \ldots, \sigma_N)$ and transition rates $w_k(\boldsymbol{\sigma})$ given in terms of the local alignment fields $h_k(\boldsymbol{\sigma})$ as

$$w_k(\boldsymbol{\sigma}) = \frac{1}{2}[1 - \sigma_k \tanh(\beta h_k(\boldsymbol{\sigma}))] \qquad h_k(\boldsymbol{\sigma}) = f_k - \sum_{j \neq k} J_{kj}\sigma_j. \tag{4}$$

Conventional demodulation [3] corresponds to taking $\hat{\sigma}_k^0 = \mathrm{sgn}(f_k)$, where $\hat{\boldsymbol{\sigma}}^0$ is our estimator for $\boldsymbol{\sigma}$, the true signal. To improve upon this, we take into account correlations induced by the spreading code. The dynamics (3) lead asymptotically to the required posterior distribution in the large t limit (i.e. the Bayesian posterior distribution). The primary performance measure for any demodulator is given by the overlap M between the signal $\boldsymbol{\sigma}^0$ and the estimate of the signal $\boldsymbol{\sigma}$, defined by

$$M(\boldsymbol{\sigma}) = \frac{1}{N} \sum_i \sigma_i \sigma_i^0. \tag{5}$$

We also use the internal energy as macroscopic order parameters

$$E(\boldsymbol{\sigma}) = \frac{1}{2N} \sum_{ij} \sigma_i J_{ij} \sigma_j = \frac{\alpha}{2} + \frac{1}{2N} \sum_{i \neq j} \sigma_i J_{ij} \sigma_j. \tag{6}$$

(note the similarity to the order parameter $r(\boldsymbol{\sigma})$ from [15]) and the contribution due to the external fields

$$F(\boldsymbol{\sigma}) = \frac{1}{N} \sum_i f_i \sigma_i. \tag{7}$$

The first-order parameter is our performance measure while the latter two give the energy from the Hamiltonian. We have chosen to split the energetic term into two, since we will find that under our assumptions both E and F will evolve according to odes containing a relaxation term and a complicated force term. Since E is quadratic in the spins and F is linear in the spins, if we took E–F (i.e. the energy of the system) to be a single order parameter, then its evolution would follow from the difference of two complicated force terms. We found that the degree of analytic complexity was the same for either choice. However, by splitting the energy into three terms, our approximation to $p_t(\boldsymbol{\sigma})$ has an extra degree of freedom and thus the approximation is better. Hence, at no extra analytic or numerical cost (compared to the standard two-order parameter theory) we obtain a better approximation to the dynamics.

Following [14–16] we may derive a Kramers–Moyal expansion for the probability density $\mathcal{P}_t(M,E,F) = \sum_{\sigma} p_t(\boldsymbol{\sigma})\delta[M - M(\boldsymbol{\sigma})]\delta[E - E(\boldsymbol{\sigma})]\delta[F - F(\boldsymbol{\sigma})]$,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}_t(M,E,F) = &-\frac{\partial}{\partial M}\left\{\mathcal{P}_t(M,E,F)\left[\left\langle\frac{1}{N}\sum_i \sigma_i^0 \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} - M\right]\right\} \\
&-\frac{\partial}{\partial E}\left\{\mathcal{P}_t(M,E,F)\left[\left\langle\frac{1}{N}\sum_i h_i^{loc}(\boldsymbol{\sigma}) \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} + \alpha - 2E\right]\right\} \\
&-\frac{\partial}{\partial F}\left\{\mathcal{P}_t(M,E,F)\left[\left\langle\frac{1}{N}\sum_i f_i(\boldsymbol{\sigma}) \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} - F\right]\right\} \\
&+\mathcal{O}\left(\frac{1}{N}\right)
\end{aligned} \tag{8}$$

where we define $h_i^{loc}(\boldsymbol{\sigma}) = \sum_{j\neq i} J_{ij}\sigma_j$. In the thermodynamic limit, on finite timescales, only the Liouville term survives in this equation, so that the order parameter triple (M, E, F) evolves deterministically according to

$$\frac{\mathrm{d}}{\mathrm{d}t}M = -M + \left\langle\frac{1}{N}\sum_i \sigma_i^0 \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -2E + \alpha + \left\langle\frac{1}{N}\sum_i h_i^{loc}(\boldsymbol{\sigma}) \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} \tag{10}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}F = -F + \left\langle\frac{1}{N}\sum_i f_i \tanh[\beta h_i(\boldsymbol{\sigma})]\right\rangle_{M,E,F;t} \tag{11}$$

where

$$\langle f(\boldsymbol{\sigma})\rangle_{M,E,F;t} = \frac{\sum_{\sigma} p_t(\boldsymbol{\sigma})\delta[M - M(\boldsymbol{\sigma})]\delta[E - E(\boldsymbol{\sigma})]\delta[F - F(\boldsymbol{\sigma})]f(\boldsymbol{\sigma})}{\sum_{\sigma} p_t(\boldsymbol{\sigma})\delta[M - M(\boldsymbol{\sigma})]\delta[E - E(\boldsymbol{\sigma})]\delta[F - F(\boldsymbol{\sigma})]}. \tag{12}$$

These flow equations (9)–(11) are still exact in the thermodynamic limit. However, to move to a practical representation, i.e. one that does not depend on the microstate probability distribution $p_t(\boldsymbol{\sigma})$, we make the assumptions underlying dynamical replica theory [14, 16]: that the observables are self-averaging with respect to the realization of the disorder and initial conditions and that we may approximate the microscopic probability distribution by the maximum entropy distribution given the values of our observables. With this approximation, the microstate probability drops out of our equations and we may then use the replica technique to remove the unpleasant fraction in (12), via

$$\frac{\sum_{\sigma}\Phi(\boldsymbol{\sigma})W(\boldsymbol{\sigma})}{\sum_{\sigma}W(\boldsymbol{\sigma})} = \lim_{n\to 0}\sum_{\boldsymbol{\sigma}^1,\ldots,\boldsymbol{\sigma}^n}\Phi(\boldsymbol{\sigma}^1)\prod_{\alpha=1}^{n}W(\boldsymbol{\sigma}^\alpha) \tag{13}$$

which we proceed to calculate in the following section.

3. Replica calculation of the flow

Using site equivalence under the disorder average, the objects we need to calculate can be expressed as

$$D(h) = \overline{\langle\sigma_1^0\delta[h - h_1(\boldsymbol{\sigma})]\rangle_{M,E,F}} \tag{14}$$

$$D(f,h) = \overline{\left\langle \delta[f - f_1]\delta\left[h - h_1^{loc}(\boldsymbol{\sigma})\right]\right\rangle_{M,E,F}} \tag{15}$$

where $\overline{\cdots}$ denotes averaging with respect to the channel noise and the spreading code. From the definition $h_1(\boldsymbol{\sigma}) \equiv f_1 - h_1^{loc}(\boldsymbol{\sigma})$, the calculation of the first distribution (14) follows quite similarly to that of the second, so we focus on the calculation of the second distribution (15) in the current section. We introduce Fourier representations for the delta functions over (M, E, F) constraining the distribution in (15) giving conjugate parameters $\{\hat{M}^\alpha, \hat{E}^\alpha, \hat{F}^\alpha\}$. To average over the disordered signal y^t, we generate its correct measure by using the partition function with the true value of the noise β_s [6]. We also write the delta functions over the fields in Fourier representation

$$\delta\left[f - \frac{1}{N}\sum_t y_t \eta_1^t\right]\delta[h - h_1(\boldsymbol{\sigma}^1)] = \int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2} \times \exp\left\{\mathrm{i}h\hat{h} + \mathrm{i}f\hat{f} - \frac{\mathrm{i}\hat{f}}{N}\sum_{t=1}^{p} y^t\eta_1^t - \frac{\mathrm{i}\hat{h}}{N}\sum_{j>1}\sum_{t=1}^{p}\eta_1^t\eta_j^t\sigma_j^1\right\}. \tag{16}$$

To get the correct scaling, we change variables $r^t = y^t/\sqrt{N}$ and then introduce

$$1 \sim \prod_{t=1}^{p}\prod_{\alpha=0}^{n}\int \mathrm{d}v^{t\alpha}\,\mathrm{d}\hat{v}^{t\alpha}\exp\left\{\mathrm{i}v^{t\alpha}\hat{v}^{t\alpha} - \frac{\mathrm{i}}{\sqrt{N}}\hat{v}^{t\alpha}\sum_{i>1}\eta_i^t\sigma_i^\alpha\right\}. \tag{17}$$

We then find

$$\begin{aligned}
D(f,h) \sim \lim_{n\to 0}\int &\frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2}\prod_\alpha[\mathrm{d}\hat{M}^\alpha\,\mathrm{d}\hat{E}^\alpha\,\mathrm{d}\hat{F}^\alpha]\prod_{t=1}^{p}\mathrm{d}r^t\prod_{t\alpha}[\mathrm{d}v^{t\alpha}\,\mathrm{d}\hat{v}^{t\alpha}] \\
&\times\exp\left(\mathrm{i}h\hat{h} + \mathrm{i}\hat{f}f + \mathrm{i}N\sum_\alpha(M\hat{M}^\alpha + E\hat{E}^\alpha)\right)\exp\left(\mathrm{i}N\sum_\alpha F\hat{F}^\alpha + \mathrm{i}\sum_{\alpha t}v^{t\alpha}\hat{v}^{t\alpha}\right. \\
&\left. - \frac{\beta_s}{2}\sum_t(r^t - v^{t0})^2 - \frac{\alpha\beta_s}{2} - \frac{\mathrm{i}}{2}\sum_\alpha\hat{E}^\alpha\sum_t v_{t\alpha}^2 - \frac{\mathrm{i}\alpha}{2}\sum_\alpha\hat{E}^\alpha - \mathrm{i}\sum_{\alpha t}\hat{F}^\alpha r^t v^{t\alpha}\right) \\
&\times\sum_{\boldsymbol{\sigma}^0,\ldots,\boldsymbol{\sigma}^n}\exp\left(-\mathrm{i}\sum_{\alpha>0}\hat{M}^\alpha\sum_i\sigma_i^\alpha\sigma_i^0\right)\sum_{\boldsymbol{\eta}}\exp\left(-\frac{\mathrm{i}}{\sqrt{N}}\sum_{t=1}^{p}\eta_1^t(\hat{f}r^t + \hat{h}v^{t1})\right. \\
&\left. - \frac{\mathrm{i}}{\sqrt{N}}\sum_{t\alpha}\hat{v}^{t\alpha}\sum_{i>1}\eta_i^t\sigma_i^\alpha\right)\exp\left(-\frac{\mathrm{i}}{\sqrt{N}}\sum_{\alpha t}\hat{F}^\alpha r^t\eta_1^t\sigma_1^\alpha\right. \\
&\left. + \frac{\beta_s}{\sqrt{N}}\sum_t\eta_1^t\sigma_1^0(r^t - v^{t0}) - \frac{\mathrm{i}}{\sqrt{N}}\sum_{t\alpha}\hat{E}^\alpha\eta_1^t\sigma_1^\alpha v^{t\alpha}\right).
\end{aligned}$$

By $\sim$ we mean that this is correct up to irrelevant normalization constants, which can be recovered later by dividing through $\overline{\langle 1\rangle_{M,E,F}}$ or requiring $\int \mathrm{d}f\,\mathrm{d}h\,D(f,h) = 1$. Performing the trace over $\boldsymbol{\eta}$ in the last line, and then expanding the resultant formulae, gives a contribution to leading order in N of

$$\exp\left\{\frac{1}{2N}\sum_t\left[\beta_s\sigma_1^0(r^t - v^{t0}) - \mathrm{i}\sum_\alpha\sigma_1^\alpha(\hat{E}^\alpha v^{t\alpha} + \hat{F}^\alpha r^t) - \mathrm{i}\hat{f}r^t - \mathrm{i}\hat{h}v^{t1}\right]^2 - \frac{1}{2N}\sum_{t,i>1,\alpha,\beta}\hat{v}^{t\alpha}\sigma_i^\alpha\hat{v}^{t\beta}\sigma_i^\beta\right\}. \tag{18}$$

We then introduce the Edwards–Anderson order parameter:

$$1 = \int \prod_{\rho<\tau} \mathrm{d}r_{\rho\tau}\,\mathrm{d}q_{\rho\tau} \exp\left(\mathrm{i}\sum_{\rho\tau} r_{\rho\tau} \sum_{i>1} \sigma_i^\rho \sigma_i^\tau - \mathrm{i}N \sum_{\rho\tau} r_{\rho\tau} q_{\rho\tau}\right) \tag{19}$$

leading to

$$\begin{aligned} D(f,h) \sim \lim_{n\to 0} \int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2} \prod_\alpha [\mathrm{d}\hat{M}^\alpha\,\mathrm{d}\hat{E}^\alpha\,\mathrm{d}\hat{F}^\alpha] \prod_{\rho<\tau} [\mathrm{d}r_{\rho\tau}\,\mathrm{d}q_{\rho\tau}]\,\mathrm{e}^{\mathrm{i}h\hat{h}+\mathrm{i}f\hat{f}} \\ \times \exp\left(\mathrm{i}N\sum_\alpha (M\hat{M}^\alpha + E\hat{E}^\alpha + F\hat{F}^\alpha) - \mathrm{i}N\sum_{\rho\tau} r_{\rho\tau}q_{\rho\tau} + (N-1)\right. \\ \times \left.\log\sum_\sigma \exp\left(\mathrm{i}\sum_{\alpha\beta} r_{\alpha\beta}\sigma^\alpha\sigma^\beta - \mathrm{i}N\sum_{\alpha>0}\hat{M}^\alpha q_{0\alpha}\right)\right) \sum_{\sigma_1} \exp\left(-\mathrm{i}\sum_\alpha \hat{M}^\alpha \sigma_1^0 \sigma_1^\alpha\right) \\ \prod_t \left\{\int \mathrm{d}r\,\mathrm{d}\mathbf{v}\,G(r,\mathbf{v}) \exp\left[\frac{1}{2N}[\beta_s\sigma_1^0(r-v^0) - \mathrm{i}\sum_\alpha \sigma_1^\alpha(\hat{E}^\alpha v^\alpha + \hat{F}^\alpha r) - \mathrm{i}\hat{f}r - \mathrm{i}\hat{h}v^1]^2\right]\right\} \end{aligned} \tag{20}$$

where

$$G(r,\mathbf{v}) = \exp\left(-\frac{1}{2}\mathbf{v}\mathbf{q}^{-1}\mathbf{v} - \frac{\beta_s}{2}(r-v^0)^2 - \frac{\mathrm{i}}{2}\sum_\alpha \hat{E}^\alpha v_\alpha^2 - \mathrm{i}\sum_\alpha \hat{F}^\alpha v^\alpha r\right). \tag{21}$$

It is instructive to view (20) as a measure over (f, h), with free parameters which are subsequently averaged over a saddle point measure. Thus, the free parameters within the $\mathcal{O}(1)$ measure take their saddle point values. The saddle point surface itself is given by

$$\begin{aligned} \Psi = \mathrm{i}\sum_\alpha (M\hat{M}^\alpha + E\hat{E}^\alpha + F\hat{F}^\alpha) + \log\sum_\sigma \exp\left(\mathrm{i}\sum_{\alpha\beta} r_{\alpha\beta}\sigma^\alpha\sigma^\beta - \mathrm{i}\sum_{\alpha\beta} r_{\alpha\beta}q_{\alpha\beta} - \mathrm{i}\sum_{\alpha>0}\hat{M}^\alpha q_{0\alpha}\right) \\ + \alpha\log\int \mathrm{d}r\,\mathrm{d}\mathbf{v} \exp\left(-\frac{1}{2}\mathbf{v}\mathbf{q}^{-1}\mathbf{v} - \frac{\beta_s}{2}(r-v^0)^2 - \frac{\mathrm{i}}{2}\sum_\alpha \hat{E}^\alpha v_\alpha^2 - \mathrm{i}\sum_\alpha \hat{F}^\alpha v^\alpha r\right). \end{aligned} \tag{22}$$

4. Replica symmetric saddle points

We make the replica symmetric assumptions, $r_{0\tau} = r, r_{\rho\tau} = R, q_{0\alpha} = m$, and $q_{\alpha\beta} = q$ and $(\hat{M}^\alpha, \hat{E}^\alpha, \hat{F}^\alpha) = (\hat{M}, \hat{E}, \hat{F})\forall\alpha$. Then, similar to the original equilibrium calculation [4–6], we have a saddle point surface

$$\begin{aligned} \Psi_{\mathrm{RS}} = \lim_{n\to 0}\frac{1}{n}\Psi = \int \mathrm{D}z \log 2\cosh(\sqrt{R}z + r) - rm - \frac{1}{2}R(1-q) + \hat{M}(M-m) + E\hat{E} + F\hat{F} \\ + \alpha\log\int \mathrm{d}r\,\mathrm{d}\mathbf{v} \exp\left\{-\frac{1}{2}\mathbf{v}\mathbf{q}^{-1}\mathbf{v} - \frac{\beta_s}{2}(r-v^0)^2 - \frac{\hat{E}}{2}\sum_\alpha v_\alpha^2 - \hat{F}\sum_\alpha v^\alpha r\right\}. \end{aligned} \tag{23}$$

We wish to evaluate the integrals I in the second line of the above so that we can take the limit $n \to 0$. It is convenient to change variables to

$$v_0 = u\sqrt{1 - \frac{m^2}{q}} - \frac{tm}{\sqrt{q}} \tag{24}$$

$$v_\alpha = z_\alpha\sqrt{1-q} - t\sqrt{q} \tag{25}$$

where u, t, $\{z_\alpha\}$ are zero mean, uncorrelated, unit variance Gaussian random variables. Then, using the shorthand $\mathrm{D}x = (2\pi)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}x^2}$ we have

$$I = \int \mathrm{D}r\,\mathrm{D}t\,\mathrm{D}u \exp\left[-\frac{\beta_s}{2}\left(u\sqrt{1-\frac{m^2}{q}} - \frac{tm}{\sqrt{q}} - r\right)^2\right] \\ \times \left\{\int \mathrm{D}z \exp\left[-\frac{\hat{E}}{2}(z\sqrt{1-q} - t\sqrt{q})^2 - \hat{F}r(z\sqrt{1-q} - t\sqrt{q})\right]\right\}^n. \tag{26}$$

Although cumbersome, these integrals are straightforward and give

$$I = \left[1 + \beta_s\left(1 - \frac{m^2}{q}\right)\right]^{-\frac{1}{2}} [1 + \hat{E}(1-q)]^{-\frac{n}{2}} \frac{1}{\sqrt{ba - c^2}} \tag{27}$$

where

$$a = \frac{\beta_s}{1 + \beta_s\left(1 - \frac{m^2}{q}\right)} - \frac{n\hat{F}^2(1-q)}{1 + \hat{E}(1-q)} \tag{28}$$

$$b = 1 + \frac{\frac{\beta_s m^2}{q}}{1 + \beta_s\left(1 - \frac{m^2}{q}\right)} + \frac{n\hat{E}q}{1 + \hat{E}(1-q)} \tag{29}$$

$$c = \frac{\beta_s \frac{m}{\sqrt{q}}}{1 + \beta_s\left(1 - \frac{m^2}{q}\right)} - \frac{n\hat{F}\sqrt{q}}{1 + \hat{E}(1-q)} \tag{30}$$

giving the replica symmetric saddle point surface

$$\Psi_{\mathrm{RS}} = \int \mathrm{D}z \log 2\cosh(\sqrt{R}z + r) - rm - \frac{1}{2}R(1-q) + \hat{M}(M-m) + E\hat{E} + F\hat{F} \\ - \frac{\alpha}{2}\left\{\log[1 + \hat{E}(1-q)] + \frac{\beta_s(\hat{E}q + 2m\hat{F} - \hat{F}^2(1-q)) - \hat{F}^2(1-q)}{\beta_s[1 + \hat{E}(1-q)]}\right\}. \tag{31}$$

Extremizing this surface we find the saddle point equations

$$r, R:\quad m = \int \mathrm{D}z \tanh(\sqrt{R}z + r) \qquad q = \int \mathrm{D}z \tanh^2(\sqrt{R}z + r) \tag{32}$$

$$\hat{M}, \hat{E}:\quad M = m \quad E = \frac{\alpha}{2}\left\{\frac{\beta_s[1 + \hat{E}(1-q)^2] - 2m\hat{F}\beta_s(1-q) + \hat{F}^2(1-q)^2(1+\beta_s)}{\beta_s[1 + \hat{E}(1-q)]^2}\right\}$$

$$\hat{F}:\quad F = \alpha\frac{\beta_s m - (1+\beta_s)\hat{F}(1-q)}{\beta_s[1 + \hat{E}(1-q)]} \tag{33}$$

$$m, q:\quad r + \hat{M} = \frac{-\alpha\hat{F}}{1 + \hat{E}(1-q)} \qquad R = \frac{\alpha\left[\hat{F}^2\left(1 + \beta_s^{-1}\right) + \hat{E}(\hat{E}q + 2m\hat{F})\right]}{[1 + \hat{E}(1-q)]^2}. \tag{34}$$

Equilibrium corresponds to $\hat{M} = 0$ and $\hat{E} = -\hat{F} = \beta$ since in this case the overlap is unconstrained while the variable conjugate to the energy is, of course, the temperature. It is a useful check to see that in this case the saddle point equations for r, R, m and q resort to their standard equilibrium expressions (as then our maximum entropy measure is just the equilibrium measure at temperature β).

4.1. Reduction of the saddle point equations

We have to express the values of the unknown conjugate-order parameters given the values of the true-order parameters. By eliminating $\hat{F}$ from the saddle point equations for F and E we obtain a quadratic equation in $\hat{E}$ for which the physical solution (utilizing the fact that in equilibrium we require $\hat{E} = \beta$) is given by

$$\hat{E}(q) = \frac{1}{2d(1-q)^2}\Big\{-[2d(1-q) + \beta_s(1+\beta_s)(1-q)^2] - \sqrt{[2d(1-q) + \beta_s(1+\beta_s)(1-q)^2]^2 - 4d(1-q)^2\big[d + \beta_s(1+\beta_s) - M^2\beta_s^2\big]}\Big\}$$

$$d = \frac{F^2\beta_s^2}{\alpha^2} - \frac{2\beta_s(1+\beta_s)E}{\alpha}. \tag{35}$$

Insertion of (35) into (33) gives us $\hat{F}(q)$, which in turn implies $R(q)$. To obtain $r(q)$ we have to use the implicit equation for r

$$M = \int \mathrm{D}z \tanh(\sqrt{R(q)}z + r). \tag{36}$$

We have resolved all free parameters into bare functions of q and hence have the relatively straightforward one-dimensional problem

$$q = \int \mathrm{D}z \tanh^2(\sqrt{R(q)}z + r(q)) \tag{37}$$

which we solve numerically.

5. Derivation of the force terms in the order parameter flow

We are now in a position to calculate $D(h)$ and $D(f, h)$. We abuse notation slightly by using $\hat{M}$, $\hat{E}$, $\hat{F}$, q when we mean their values taken in the saddle point given the values of M, E and F rather than seeing them as variables. With that taken into account, we may write

$$D(h) = \lim_{n\to 0} \int \frac{\mathrm{d}\hat{h}}{2\pi} \mathrm{e}^{\mathrm{i}h\hat{h}} \sum_{\boldsymbol{\sigma}} \sigma_0 \exp\left(-\hat{M}\sum_{\alpha>0}\sigma^0\sigma^\alpha + \frac{\alpha}{2}\Xi(\boldsymbol{\sigma})\right) \tag{38}$$

$$D(f, h) = \lim_{n\to 0} \int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2} \mathrm{e}^{\mathrm{i}h\hat{h}+\mathrm{i}f\hat{f}} \sum_{\boldsymbol{\sigma}} \exp\left(-\hat{M}\sum_{\alpha>0}\sigma^0\sigma^\alpha + \frac{\alpha}{2}\Lambda(\boldsymbol{\sigma})\right) \tag{39}$$

$$\Xi(\boldsymbol{\sigma}) = \frac{\int \mathrm{d}r\,\mathrm{d}\mathbf{v} G(r, \mathbf{v})\big[\beta_s(r - v^0)\sigma^0 - \sum_\alpha \sigma^\alpha(\hat{F}r + \hat{E}v_\alpha) - \mathrm{i}\hat{h}(r - v^1)\big]^2}{\int \mathrm{d}r\,\mathrm{d}\mathbf{v} G(r, \mathbf{v})} \tag{40}$$

$$\Lambda(\boldsymbol{\sigma}) = \frac{\int \mathrm{d}r\,\mathrm{d}\mathbf{v} G(r, \mathbf{v})\big[\beta_s(r - v^0)\sigma^0 - \sum_\alpha \sigma^\alpha(\hat{F}r + \hat{E}v_\alpha) - \mathrm{i}\hat{h}v^1 - \mathrm{i}\hat{f}r\big]^2}{\int \mathrm{d}r\,\mathrm{d}\mathbf{v} G(r, \mathbf{v})} \tag{41}$$

$$G(r, \mathbf{v}) = \exp\left(-\frac{1}{2}\mathbf{v}\mathbf{q}^{-1}\mathbf{v} - \frac{\beta_s}{2}(r - v^0)^2 - \frac{\hat{E}}{2}\sum_\alpha v_\alpha^2 - \hat{F}\sum_\alpha v_\alpha r\right). \tag{42}$$

5.1. Evolution of the overlap

We concentrate first on Ξ. It is convenient to introduce the shorthand for averages over the measure (42),

$$\langle\cdots\rangle_G = \frac{\int \mathrm{d}r\,\mathrm{d}\mathbf{v}G(r,\mathbf{v})\ldots}{\int \mathrm{d}r\,\mathrm{d}\mathbf{v}G(r,\mathbf{v})} \tag{43}$$

with which we define the shorthand:

$$g_{10} = \alpha\beta_s\langle(r-v^0)(r-v^1)\rangle_G \qquad g_{0\alpha} = \alpha\beta_s\langle(r-v^0)(\hat{F}r+\hat{E}v^\alpha)\rangle_G + \hat{M} \tag{44}$$

$$g_{11} = \alpha\langle(r-v^1)^2\rangle_G \qquad g_{\alpha\beta} = \alpha\langle(\hat{F}r+\hat{E}v^\alpha)(\hat{F}r+\hat{E}v^\beta)\rangle_G \tag{45}$$

$$g_{1\alpha} = \alpha\langle(r-v^1)(\hat{F}r+\hat{E}v^\alpha)\rangle_G \quad \forall\alpha>1 \qquad g_{111} = \alpha\langle(r-v^1)(\hat{F}r+\hat{E}v^1)\rangle_G. \tag{46}$$

We calculate these factors in appendix A, for now we merely note that $g_{\alpha\beta} = R$ while $-g_{0\alpha} = r$. It is possible to ignore constants such as $\beta_s^2\langle(r-v_0)^2\rangle_G$ since they may be dealt with via overall normalization of the measure $D(h)$. Then

$$D(h) \sim \lim_{n\to 0}\int\frac{\mathrm{d}\hat{h}}{2\pi}\exp\left(-\frac{\hat{h}^2 g_{11}}{2}+\mathrm{i}h\hat{h}\right)\sum_{\sigma}\sigma_0\exp\left(\mathrm{i}\hat{h}[\sigma^1 g_{111}+\sum_{\alpha>1}\sigma^\alpha g_{1\alpha}-\sigma_0 g_{10}]\right)$$
$$\times\int \mathrm{D}x\exp\left(x\sqrt{R}\sum_\alpha\sigma^\alpha+\sigma_0\sum_\alpha\sigma^\alpha r\right).$$

Since we only require $D(h)$ in a term of the form $\int \mathrm{d}h\,D(h)\tanh(\beta h)$ it is possible to make the gauge transformation $x, h, \hat{h} \to \sigma_0 x, \sigma_0 h, \sigma_0\hat{h}$ to remove the dependence on σ_0. We then perform the trace over the other spins to obtain

$$D(h) \sim \lim_{n\to 0}\int\frac{\mathrm{d}\hat{h}}{2\pi}\exp\left(-\frac{\hat{h}^2 g_{11}}{2}+\mathrm{i}h\hat{h}\right)\int \mathrm{D}x\exp(-\mathrm{i}\hat{h}g_{10})$$
$$\times\cosh[\mathrm{i}\hat{h}g_{111}+x\sqrt{R}+r]\cosh^{n-1}[\mathrm{i}\hat{h}g_{1\alpha}+x\sqrt{R}+r].$$

In order to move our integration contour so that we could perform these integrals, we expand the first cosh function,

$$\cosh[\mathrm{i}\hat{h}g_{111}+x\sqrt{R}+r] = \cosh(\mathrm{i}\hat{h}\Delta)\cosh[\mathrm{i}\hat{h}g_{1\alpha}+x\sqrt{R}+r]$$
$$+\sinh(\mathrm{i}\hat{h}\Delta)\sinh[\mathrm{i}\hat{h}g_{1\alpha}+x\sqrt{R}+r] \tag{47}$$

where $\Delta = g_{111} - g_{1\alpha}$. The integral over the contribution to $D(h)$ containing only cosh functions is relatively straightforward and gives

$$\frac{1}{2\sqrt{2\pi g_{11}}}\left\{\exp\left[-\frac{(h-g_{10}+\Delta)^2}{2g_{11}}\right]+\exp\left[-\frac{(h-g_{10}-\Delta)^2}{2g_{11}}\right]\right\}. \tag{48}$$

For the term containing the sinh functions from (47), we shift the integration variable $x \to x - \mathrm{i}\hat{h}\frac{g_{1\alpha}}{\sqrt{R}}$ and take the limit $n \to 0$ to obtain,

$$\int\frac{\mathrm{d}\hat{h}}{2\pi}\exp\left(-\frac{\hat{h}^2}{2}\left(g_{11}-\frac{g_{1\alpha}^2}{R}\right)+\mathrm{i}h\hat{h}-\mathrm{i}\hat{h}g_{10}\right)\sinh(\mathrm{i}\hat{h}\Delta)\int \mathrm{D}x\exp\left(\mathrm{i}x\hat{h}\frac{g_{1\alpha}}{\sqrt{R}}\right)\tanh(x\sqrt{R}+r).$$

We proceed by writing $\sinh(\mathrm{i}\hat{h}\Delta) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}\hat{h}\Delta}-\mathrm{e}^{-\mathrm{i}\hat{h}\Delta})$ and we first treat the term with containing $\mathrm{e}^{\mathrm{i}\hat{h}\Delta}$, integrating over $\hat{h}$:

$$\int\frac{\mathrm{d}\hat{h}}{4\pi}\exp\left(-\frac{\hat{h}^2}{2}\left(g_{11}-\frac{g_{1\alpha}^2}{R}\right)+\hat{h}\mathrm{i}(h-g_{10}+\Delta)\right)\int \mathrm{D}x\exp\left(\mathrm{i}x\hat{h}\frac{g_{1\alpha}}{\sqrt{R}}\right)\tanh(x\sqrt{R}+r)$$

$$= \frac{1}{2\sqrt{2\pi\alpha\left(g_{11} - \frac{g_{1\alpha}^2}{R}\right)}} \int \mathrm{D}x \exp\left[-\frac{1}{2\left(g_{11} - \frac{g_{1\alpha}^2}{g_{\alpha\beta}}\right)}\left(h - g_{10} + \Delta + x\frac{g_{1\alpha}}{\sqrt{R}}\right)^2\right] \times \tanh(x\sqrt{R} + r). \tag{49}$$

By a careful change of variables this reduces to

$$\frac{1}{\sqrt{2\pi g_{11}}} \mathrm{e}^{-\frac{h^2}{2g_{11}}} \int \mathrm{D}x \tanh\left[x\sqrt{\frac{Rg_{11} - g_{1\alpha}^2}{g_{11}}} - \frac{g_{1\alpha}}{g_{11}}h + r\right]. \tag{50}$$

The case with $\mathrm{e}^{-\mathrm{i}\Delta\hat{h}}$ follows identically, so finally we find that

$$D(h) = \frac{\mathrm{e}^{-\frac{1}{2g_{11}}(h-g_{10}+\Delta)}}{\sqrt{2\pi g_{11}}}\left\{1 + \int \mathrm{D}x \tanh\left[x\sqrt{\frac{\left(Rg_{11} - g_{1\alpha}^2\right)}{g_{11}}} - \frac{g_{1\alpha}}{g_{11}}(h - g_{10} + \Delta) + r\right]\right\} + \frac{\mathrm{e}^{-\frac{1}{2g_{11}}(h-g_{10}-\Delta)}}{\sqrt{2\pi g_{11}}}\left\{1 - \int \mathrm{D}x \tanh\left[x\sqrt{\frac{\left(Rg_{11} - g_{1\alpha}^2\right)}{g_{11}}} - \frac{g_{1\alpha}}{g_{11}}(h - g_{10} - \Delta) + r\right]\right\}. \tag{51}$$

We wish to consider the term $\int \mathrm{d}h\, D(h)\tanh(\beta h)$, which is required to calculate the force term in the differential equation (9). This can be most easily effected by first making the transformation $h \to h + g_{10} \mp \Delta$ as required, followed by $h \to \sqrt{g_{11}}h$. We then have

$$\int \mathrm{d}h\, D(h)\tanh(\beta h) = \frac{1}{2}\int \mathrm{D}h\,\mathrm{D}x\left\{1 + \tanh\left[x\sqrt{\frac{Rg_{11} - g_{1\alpha}^2}{g_{11}}} - \frac{g_{1\alpha}h}{\sqrt{g_{11}}} + r\right]\right\}\tanh[\beta(\sqrt{g_{11}}h + g_{10} - \Delta)] \tag{52}$$

$$+ \frac{1}{2}\int \mathrm{D}h\,\mathrm{d}x\left\{1 - \tanh\left[x\sqrt{\frac{Rg_{11} - g_{1\alpha}^2}{g_{11}}} - \frac{g_{1\alpha}h}{\sqrt{g_{11}}} + r\right]\right\}\tanh[\beta(\sqrt{g_{11}}h + g_{10} + \Delta)]. \tag{53}$$

We then rotate the Gaussian integration variables which leads to our final result

$$\int \mathrm{d}h\, D(h)\tanh(\beta h) = \frac{1}{2}\int \mathrm{D}u\,\mathrm{D}v\{1 + \tanh[\sqrt{R}u + r]\}\tanh\left[\beta\left(\sqrt{\frac{Rg_{11} - g_{1\alpha}^2}{R}}v - \frac{g_{1\alpha}}{\sqrt{R}}u + g_{10} - \Delta\right)\right] + \frac{1}{2}\int \mathrm{D}h\,\mathrm{D}x\{1 - \tanh[\sqrt{R}u + r]\}\tanh\left[\beta\left(\sqrt{\frac{Rg_{11} - g_{1\alpha}^2}{R}}v - \frac{g_{1\alpha}}{\sqrt{R}}u + g_{10} + \Delta\right)\right]. \tag{54}$$

5.2. Fixed points for the overlap in equilibrium

As a useful test of our analysis thus far, we show that in equilibrium the equilibrium value of the overlap is a fixed point of our dynamics. In equilibrium we have $\hat{M} = 0$ and $\hat{E} = -\hat{F} = \beta$ as argued above. So we can read off the equilibrium values of our factors $g_{\ldots}$ from appendix A, in particular $-\beta g_{1\alpha} = R$ and $\beta g_{10} = r$, while $\beta g_{1\alpha} = -R$ and $\beta g_{11} = -g_{111}$, whence

$$\beta\Delta = \beta(g_{111} - g_{1\alpha}) = R - \beta^2 g_{11} \tag{55}$$

and so

$$\int \mathrm{d}h\, D(h)\tanh(\beta h) = \int \mathrm{D}u \tanh[\sqrt{R}u + r] \tag{56}$$

according to the identity

$$\tanh(u) = \frac{1}{2}[1 - \tanh(u)] \int \mathrm{D}y \tanh(u + yz - z^2) + \frac{1}{2}[1 + \tanh(u)] \int \mathrm{D}y \tanh(u + yz + z^2) \tag{57}$$

of [15]. Since our saddle point equations (32)–(34) for m, q, R, r in equilibrium are equivalent to the equilibrium saddle point equations, we see that one fixed point of our differential equation for the overlap (9) is given by the equilibrium overlap.

5.3. Evolution of the energetic terms E and F

We now turn to $D(f, h)$, for which we need the evaluation of $\Lambda(\boldsymbol{\sigma})$ as defined in (41). On top of our previous definitions we now introduce

$$g_{rr} = \alpha\langle r^2\rangle_G \qquad g_{vv} = \alpha\langle v_1^2\rangle_G \tag{58}$$

$$g_{rv} = \alpha\langle r v_1\rangle_G \qquad g_{ra} = \alpha\langle r(\hat{F}r + \hat{E}v_\alpha)\rangle_G \tag{59}$$

$$g_{va} = \alpha\langle v_1(\hat{F}r + \hat{E}v_\alpha)\rangle_G \quad \forall\alpha > 1 \qquad g_{v1} = \alpha\langle v_1(\hat{F}r + \hat{E}v_1)\rangle_G \tag{60}$$

$$g_{0r} = \alpha\beta_s\langle r(r - v_0)\rangle_G \qquad g_{0v} = \alpha\beta_s\langle v_1(r - v_0)\rangle_G \tag{61}$$

which are also calculated in appendix A. Using these g factors we may write

$$D(f, h) \sim \lim_{n\to 0} \int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2} \exp\left(-\frac{\hat{h}^2 g_{vv}}{2} - \frac{\hat{f}^2 g_{rr}}{2} - \hat{f}\hat{h}g_{rv} + \mathrm{i}h\hat{h} + \mathrm{i}f\hat{f}\right) \times \sum_{\boldsymbol{\sigma}} \exp\left(\mathrm{i}\hat{h}\left[\sigma^1 g_{v1} + \sum_{\alpha>1}\sigma^\alpha g_{v\alpha} - \sigma_0 g_{0v}\right]\right)$$
$$\int \mathrm{D}x \exp\left(x\sqrt{R}\sum_\alpha \sigma^\alpha + r\sigma_0\sum_\alpha\sigma^\alpha\right)\exp\left(-\mathrm{i}\hat{f}\left[\sigma_0 g_{r0} - \sum_\alpha \sigma_\alpha g_{r\alpha}\right]\right) \tag{62}$$

which we may rewrite as

$$D(f, h) \sim \lim_{n\to 0} \int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{(2\pi)^2} \exp\left(-\frac{\hat{h}^2 g_{vv}}{2} - \frac{\hat{f}^2 g_{rr}}{2} - \hat{f}\hat{h}g_{rv} + \mathrm{i}h\hat{h} + \mathrm{i}f\hat{f}\right) \times \int \mathrm{D}x \sum_{\sigma^0} \exp(-\mathrm{i}\sigma_0(\hat{h}g_{0v} + \hat{f}g_{0r}))\cosh[\mathrm{i}\hat{h}g_{v1} + \mathrm{i}\hat{f}g_{r\alpha} + x\sqrt{R} + \sigma^0 r] \times \cosh^{n-1}[\mathrm{i}\hat{h}g_{v\alpha} + \mathrm{i}\hat{f}g_{r\alpha} + x\sqrt{R} + \sigma^0 r]. \tag{63}$$

Following a procedure similar to that used in section 5.1 we write

$$\cosh[\mathrm{i}\hat{h}g_{v1} + \mathrm{i}\hat{f}g_{r\alpha} + x\sqrt{R} + \sigma^0 r] = \cosh[-\mathrm{i}\hat{h}\Delta]\cosh[\mathrm{i}\hat{h}g_{v\alpha} + \mathrm{i}\hat{f}g_{r\alpha} + x\sqrt{R} + \sigma^0 r] + \sinh[-\mathrm{i}\hat{h}\Delta]\sinh[\mathrm{i}\hat{h}g_{v\alpha} + \mathrm{i}\hat{f}g_{r\alpha} + x\sqrt{R} + \sigma^0 r] \quad (64)$$

since $g_{v1} - g_{v\alpha} = -\Delta$. It is convenient to use the change of variable $h \to h + g_{0v}\sigma_0 \pm \alpha\Delta$ and $f \to f + g_{0r}\sigma_0$. The terms in (63) containing the cosh terms from (64) can be treated in a manner similar to section 5.1 and contribute

$$\frac{\sum_{\sigma_0}}{4\pi\sqrt{g_{rr}g_{vv} - g_{rv}^2}} \exp\left(-\frac{g_{rr}h^2}{2(g_{rr}g_{vv} - g_{rv}^2)} - \frac{g_{vv}f^2}{2(g_{rr}g_{vv} - g_{rv}^2)} + \frac{g_{rv}fh}{(g_{rr}g_{vv} - g_{rv}^2)}\right). \quad (65)$$

Now we tackle the more tricky sinh terms from (63) which requires the shift of the x integral in the complex plane but then gives

$$\begin{aligned}
&\sum_{\sigma_0}\int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{2(2\pi)^2}\exp\left(-\frac{\hat{h}^2 g_{vv}}{2} - \frac{\hat{f}^2 g_{rr}}{2} - \hat{f}\hat{h}g_{rv} + \mathrm{i}h\hat{h} + \mathrm{i}f\hat{f}\right) \\
&\qquad \times \int \mathrm{D}x \tanh[x\sqrt{R} + \mathrm{i}(\hat{h}g_{v\alpha} + \hat{f}g_{r\alpha}) + r\sigma_0] \\
&\qquad = \sum_{\sigma_0}\int \frac{\mathrm{d}\hat{h}\,\mathrm{d}\hat{f}}{2(2\pi)^2}\exp\left(-\frac{\hat{h}^2}{2}\left(g_{vv} - \frac{g_{v\alpha}^2}{R}\right) - \frac{\hat{f}^2}{2}\left(g_{rr} - \frac{g_{r\alpha}^2}{R}\right)\right. \\
&\qquad -\hat{f}\hat{h}\left(g_{rv} - \frac{g_{r\alpha}g_{v\alpha}}{R}\right) + \mathrm{i}\hat{h}\left(h + \frac{xg_{v\alpha}}{\sqrt{R}}\right) \\
&\qquad \left. +\mathrm{i}\hat{f}\left(f + \frac{xg_{r\alpha}}{\sqrt{R}}\right)\right) \times \int \mathrm{D}x \tanh[x\sqrt{R} + r\sigma_0].
\end{aligned} \quad (66)$$

Proceeding with the integrals over $\hat{h}$, $\hat{f}$ and after some rearrangement we find,

$$\begin{aligned}
&\sum_{\sigma_0}\frac{1}{2\pi\sqrt{g_{rr}g_{vv} - g_{rv}^2}}\exp\left(-\frac{g_{rr}h^2}{2(g_{rr}g_{vv} - g_{rv}^2)} - \frac{g_{vv}f^2}{2(g_{rr}g_{vv} - g_{rv}^2)} + \frac{g_{rv}fh}{(g_{rr}g_{vv} - g_{rv}^2)}\right) \\
&\qquad \times \int \mathrm{D}x \tanh\left[x\sqrt{\frac{R}{C}} - \frac{(g_{rr}g_{v\alpha} - g_{r\alpha}g_{rv})}{(g_{rr}g_{vv} - g_{rv}^2)}h - \frac{(g_{vv}g_{r\alpha} - g_{v\alpha}g_{rv})}{(g_{rr}g_{vv} - g_{rv}^2)}f + r\sigma_0\right].
\end{aligned} \quad (67)$$

Putting this all together we have

$$\begin{aligned}
&\int \mathrm{d}f\,\mathrm{d}h D(f,h) f \tanh[\beta(f-h)] \\
&\qquad = \sum_{\sigma_0\sigma_1}\int \frac{\mathrm{d}f\mathrm{d}h}{4\pi\sqrt{g_{rr}g_{vv} - g_{rv}^2}} \\
&\qquad \times \exp\left(-\frac{g_{rr}h^2}{2(g_{rr}g_{vv} - g_{rv}^2)} - \frac{g_{vv}f^2}{2(g_{rr}g_{vv} - g_{rv}^2)} + \frac{g_{rv}fh}{(g_{rr}g_{vv} - g_{rv}^2)}\right) \\
&\qquad \times \left\{1 + \sigma_1\int \mathrm{D}x \tanh\left[x\sqrt{\frac{R}{C}} - \frac{(g_{rr}g_{v\alpha} - g_{r\alpha}g_{rv})}{(g_{rr}g_{vv} - g_{rv}^2)}h\right.\right. \\
&\qquad \left.\left. - \frac{(g_{vv}g_{r\alpha} - g_{v\alpha}g_{rv})}{(g_{rr}g_{vv} - g_{rv}^2)}f + r\sigma_0\right]\right\}(f + \sigma_0 g_{0r}) \\
&\qquad \times \tanh[\beta(f + g_{0r}\sigma_0 - (h + g_{0v}\sigma_0 + \sigma_1\Delta))].
\end{aligned} \quad (68)$$

Rescaling our Gaussian variables in order to write the measures as standard Gaussian measures we finally have the force term required for (11)

$$\int \mathrm{d}f\,\mathrm{d}h\,D(f,h)\,f\tanh[\beta(f-h)] \\ = \frac{1}{4}\sum_{\sigma_0\sigma_1}\int \mathrm{D}y\,\mathrm{D}z\left[1+\sigma_1\int \mathrm{D}x\tanh(A+B)\right]C\tanh[\beta(C-D)] \tag{69}$$

with

$$A = x\sqrt{\frac{R\left(g_{rr}g_{vv}-g_{rv}^2\right)-g_{v\alpha}^2 g_{rr}-g_{r\alpha}^2 g_{vv}+2g_{r\alpha}g_{v\alpha}g_{rv}}{\left(g_{rr}g_{vv}-g_{rv}^2\right)}} \tag{70}$$

$$B = r\sigma_0 - \sqrt{g_{rr}^{-1}}\left[\frac{g_{rr}g_{v\alpha}-g_{r\alpha}g_{rv}}{\sqrt{g_{rr}g_{vv}-g_{rv}^2}}+g_{r\alpha}y\right] \tag{71}$$

$$C = \sqrt{g_{rr}}\,y + g_{0r}\sigma_0 \tag{72}$$

$$D = \sqrt{\frac{\left(g_{rr}g_{vv}-g_{rv}^2\right)}{g_{rr}}}\,z + \sqrt{\frac{g_{rv}^2}{g_{rr}}}\,y + g_{0v}\sigma_0 + \Delta\sigma_1 \tag{73}$$

while with these definitions, the force term required for (10) is

$$\int \mathrm{d}f\,\mathrm{d}h\,D(f,h)\,h\tanh[\beta(f-h)] \\ = \frac{1}{4}\sum_{\sigma_0\sigma_1}\int \mathrm{D}y\,\mathrm{D}z\left[1+\sigma_1\int \mathrm{D}x\tanh(A+B)\right]D\tanh[\beta(C-D)]. \tag{74}$$

6. Order parameter flow and comparison with simulations

We now have a closed set of equations describing deterministic order parameter flow. They are

$$\frac{\mathrm{d}}{\mathrm{d}t}M = -M + \int \mathrm{d}h\,D(h)\tanh(\beta h) \tag{75}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}E = -2E + \alpha + \int \mathrm{d}f\,\mathrm{d}h\,D(f,h)\,h\tanh[\beta(f-h)] \tag{76}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}F = -F + \int \mathrm{d}f\,\mathrm{d}h\,D(f,h)\,f\tanh[\beta(f-h)] \tag{77}$$

where $D(h)$ and $D(f,h)$ at any instant depend on the triple (M, E, F) (as well as the statistics of the quenched disorder via β_s and α).

There are obviously a variety of initial conditions with which we could start. However, two are of particular importance from a practical point of view, since they are easily accessible computationally. The first is the random initial state (i.e. $\sigma_i(0) = \pm 1$ with probability $\frac{1}{2}$). It is straightforward to derive the initial states of our order parameters as

$$M_{\mathrm{RIS}} = 0 \qquad F_{\mathrm{RIS}} = 0 \qquad E_{\mathrm{RIS}} = \frac{\alpha}{2}. \tag{78}$$

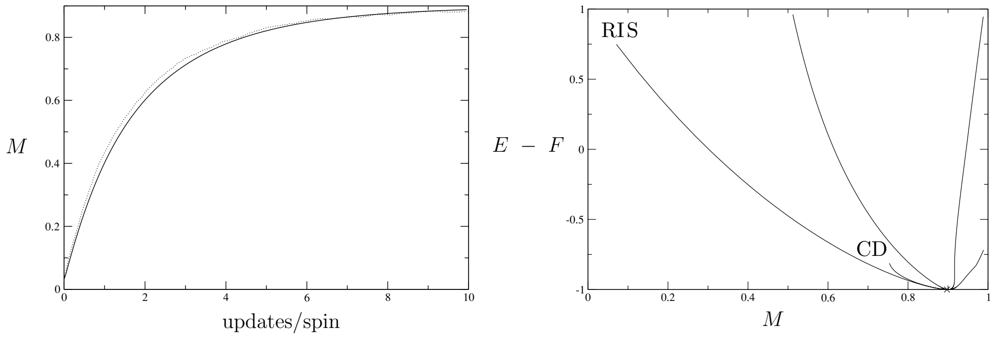

Figure 1. Order parameter flow for MPM decoding with $\alpha = \beta = \beta_s = 2$. Left: we compare the results for the overlap M of solving our order parameter equations numerically (solid line) with Monte Carlo simulations (dotted line). The Monte Carlo simulations are performed with system size 2 000 averaged over 50 samples (the standard deviation is less than 10^{-3}). Right: we examine the flow in the energy (E–F)-overlap (M) plane taking as initial conditions the random initial state (RIS), the conventional demodulator (CD) and several other (non-practical) initial conditions. The cross marks the equilibrium solution. Note that flow is from top to bottom as the dynamics serves to lower the energy of the system.

A second important initial state is that given by the conventional demodulator [3], with $\sigma_i(0) = \mathrm{sgn}(f_i)$. We derive the values of our order parameters in appendix B, which are

$$M_{CD} = \mathrm{Erf}\left[\sqrt{\frac{\alpha}{2\left(1+\beta_s^{-1}\right)}}\right] \qquad F_{CD} = \int \mathrm{d}z \left|\alpha + \sqrt{\alpha\left(1+\beta_s^{-1}\right)}z\right| \tag{79}$$

$$E_{CD} = \frac{\alpha}{2}\left[1 + 2M_{CD}\chi + \chi^2\left(1+\beta_s^{-2}\right)\right] \qquad \chi = \sqrt{\frac{2}{\pi\alpha\left(1+\beta_s^{-1}\right)}}\exp\left(-\frac{\alpha}{2\left(1+\beta_s^{-1}\right)}\right) \tag{80}$$

where $\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \mathrm{d}t\, \mathrm{e}^{-t^2}$.

As an initial basic test of our theory, we compare the flow described by the solution of our order parameter equations to Monte Carlo simulations of the original Glauber dynamics in figure 1 (left). The temperature is relatively high compared to realistic values but we see that at least in this regime we have excellent agreement between the theory and simulations, justifying our assumptions and validating our method. To visualize the flow in order parameter space, we have plotted the overlap M versus the energy E–F in figure 1 (right). We have started from the RIS initial condition, the CD initial condition as well as several other cases. The latter are not practical in the sense that we have no algorithm that can actually find these initial conditions, but they do give an idea of the shape of flow within phase space. This also highlights an advantage of the theoretical approach we have presented—these other flows are not observable from pure Monte Carlo simulations.

6.1. Phase coexistence and basins of attractions

It was noted by Tanaka [4, 5] that for certain parameter regimes a spinodal would be encountered. This has drastic implications for the decoding problem, since if there are metastable states with a high bit error rate, local search algorithms will fail to find the low bit error rate solution. To go beyond these arguments, a dynamic approach is required and we have now provided one. It is possible to look at flow in parameter space and see explicitly

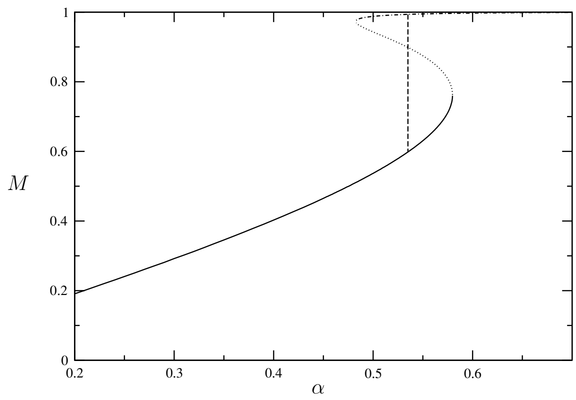

Figure 2. The equilibrium value of the overlap M versus as a function of α for MPM decoding $\beta = \beta_s = 17$. For $\alpha \in [0.482, \ldots, 0.5798\ldots]$ there is phase coexistence with the poor solution given by the solid line, the unstable solution given by the dotted line and the good solution given by the dot-dashed line. The thermodynamic transition between the good solution and the poor solution is at $\alpha \approx 0.535\,25$.

the basins of attraction for both the good and bad solutions. Spinodals occur for both MAP decoding and MPM decoding. We focus on the latter since the MPM decoding is optimal in terms of the bit error rate. Since it is at Nishimori's temperature, for equilibrium states at least, there are no complications due to replica symmetry breaking (we cannot guarantee this for the dynamical saddle point at present, but we hope to investigate this further in a later work).

In figure 2 we show a plot of the equilibrium overlap against α for $\beta = \beta_s = 17$. It is of some interest to see which of the stable solutions the decoding algorithm converges to past the spinodal point where we have coexistence.

In figure 3 we plot the flow of the dynamics in the energy-overlap plane. We see that as expected the random initial state and conventional demodulator state both flow into the poor attractor. The agreement between theory and simulations is quite reasonable up to some low energy point where we believe replica symmetry breaking (RSB) starts to play an important role in the simulations (although our current replica symmetric theory is unable to take it into account). We also note that starting from the CD state the overlap gets worse before improving to a state better than the CD in the theoretical approach, while in the simulations, it never recovers its initial overlap. Thus, starting from CD and running a local search algorithm can significantly decrease performance levels in a practical setting. We also see that the basin of attraction for the good final state is significantly smaller than that for the poor final state, the boundary is well over half way between the two states in $(E–F, M)$ space. This also shows how distance in $(E–F, M)$ space is an unreliable guide to the direction of flow towards attractors.

In figure 4 we examine the flow near the spinodal point with high α to see whether it is practically possible to attain the good solution when the poor solution is present. We see that at this low temperature we can qualitatively describe the flow as first going to low energy states then increasing the overlap. Although it cannot be seen from these pictures, what happens is that the energy is lowered relatively quickly while the increase in overlap takes some time. In any case, since we have the overlap increasing while in a low energy state, the flow goes past the poor fixed point (if it is there). Thus we conclude that there is no currently available practical initial state that will lead to good decoding past the spinodal point.

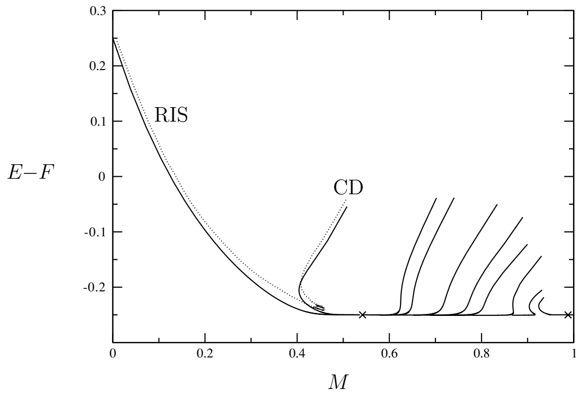

Figure 3. We examine the flow through phase space projected onto the energy (E–F)-overlap (M) plane. We both solve our order parameter equations numerically (solid line) and compare against Monte Carlo simulations (dotted line) for the two important initial states, CD and RIS. We work with MPM decoding at $\beta_s = \beta = 17$ and $\alpha = 0.5$. The crosses mark the two stable solutions to the equilibrium problem. The Monte Carlo simulations are performed with system size 2000 averaged over 50 samples. We ran both simulations and our theory for 1000 updates per spin. We see that for this parameter regime there is phase coexistence. The line labelled RIS starts from a random initial state, while the one labelled CD has the conventional demodulator as its initial state. Both flow into the poor attractor. We also show flow starting from some other initial states to get a better idea of the basins of attraction for the two phases. Note that all flow is from top to bottom as the dynamics lowers the energy of the system.

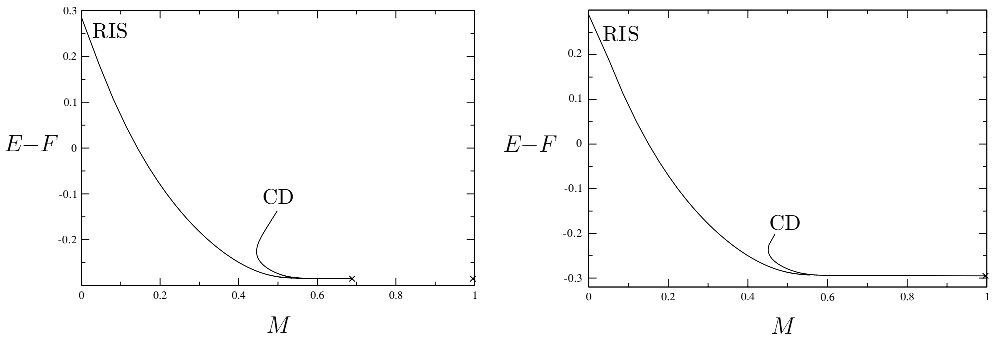

Figure 4. Order parameter flow for MPM decoding with $\beta = \beta_s = 17$ either side of the spinodal point from RIS and CD initial conditions. The crosses mark fixed points. Left: order parameter flow for $\alpha = 0.57$ where both good and bad fixed points exist. Right: order parameter flow for $\alpha = 0.59$ where only the good solution exists.

7. Conclusions

CDMA is an important standard used in modern mobile communications. Tools from statistical physics have provided and will continue to provide useful ways of examining the detection problem. Here we have developed and used dynamical replica theory to study the dynamics of the detection problem for a prototypical local search algorithm, namely the Gibbs sampler under Glauber dynamics. Although this approach is only an analytic approximation, it provides a useful counterpart to both density evolution and generating functional analysis as a tool for examining dynamic rather than equilibrium properties. As we have seen in comparison with Monte Carlo simulations the approximation is a reasonably good one. We have also calculated the basin of attraction for a particular set of parameters, in the region where there is phase coexistence. As expected, we have seen that the practically available initial states, CD and RIS, both flow to the poor solution, when the poor solution exists. We have also seen that in this case the overlap decreases initially from the CD state with our search algorithm and that the basin of attraction for the good solution is relatively small in energy-overlap space. One obvious extension of this work would be to increase the order parameter set to improve the level of approximation. Since the number of updates per spin required to visit interesting regimes is of the order of 10^3, this would have to be done in a way that was compatible with practical numerical solution. Another interesting problem is examining the role of replica symmetry breaking on the dynamics. Finally, a project that we are already working on is using the dynamical replica approach for parallel update dynamics where we can compare its predictions to the exact theory of generating functional analysis [12].

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Appendix A. Calculation of factors defining $D(h)$ and $D(f,h)$

The various factors defined in (44)–(46) and (58)–(61) are all simple combinations of

$$\langle r^2\rangle_G \qquad \langle rv^0\rangle_G \qquad \langle rv^\alpha\rangle_G \qquad \langle v^0v^\alpha\rangle_G \qquad \langle v^\alpha v^\beta\rangle_G \qquad \langle v^\alpha v^\alpha\rangle_G \tag{A.1}$$

which are moments of the measure $G(r,\mathbf{v})$ (42). The calculation is not dissimilar to that carried out in the equilibrium calculation; we have to just be careful with the algebra over various Gaussian integrals. We find,

$$\langle v^\alpha v^\alpha\rangle_G = \frac{1-q}{1+\hat{E}(1-q)} + \frac{q+(1-q)^2\hat{F}^2\left(1+\beta_s^{-1}\right)-2(1-q)\hat{F}m}{[1+\hat{E}(1-q)]^2} \tag{A.2}$$

$$\langle r^2\rangle_G = \left(1+\beta_s^{-1}\right) \tag{A.3}$$

$$\langle rv^0\rangle_G = 1 \tag{A.4}$$

$$\langle rv^\alpha\rangle_G = \frac{m-(1-q)\hat{F}\left(1+\beta_s^{-1}\right)}{[1+\hat{E}(1-q)]} \tag{A.5}$$

$$\langle v^0v^\alpha\rangle_G = \frac{m-(1-q)\hat{F}}{[1+\hat{E}(1-q)]} \tag{A.6}$$

$$\langle v^\alpha v^\beta\rangle_G = \frac{q+(1-q)^2\hat{F}^2\left(1+\beta_s^{-1}\right)-2(1-q)\hat{F}m}{[1+\hat{E}(1-q)]^2}. \tag{A.7}$$

Hence the various factors are simply

$$g_{10} = \alpha\frac{1+(1-q)(\hat{E}+\hat{F})}{[1+\hat{E}(1-q)]} \qquad g_{0\alpha} = r \qquad g_{\alpha\beta} = R \tag{A.8}$$

and

$$g_{11} = \alpha \frac{\beta_s^{-1} + 2 - 2m + \hat{E}(1-q)^2 + 2(1-q)(\hat{E}+\hat{F})\left(\beta_s^{-1} + 1 - m\right) + \left(\beta_s^{-1}+1\right)(1-q)^2(\hat{E}+\hat{F})^2}{[1+\hat{E}(1-q)]^2}.$$

We then have

$$g_{1\alpha} = \alpha \frac{\left(1+\beta_s^{-1}\right)\hat{F} + (\hat{E}-\hat{F})m - \hat{E}q + (\hat{E}+\hat{F})(1-q)\left(\hat{E}m + \hat{F}\left(1+\beta_s^{-1}\right)\right)}{[1+\hat{E}(1-q)]^2} \tag{A.9}$$

which is $-R$ in equilibrium. Then

$$\Delta = \frac{-\alpha\hat{E}(1-q)}{1+\hat{E}(1-q)} \qquad g_{v\alpha} = \alpha \frac{\hat{F}m + \hat{E}q - (1-q)\left[m\hat{E}\hat{F} + \hat{F}^2\left(1+\beta_s^{-1}\right)\right]}{[1+\hat{E}(1-q)]^2} \tag{A.10}$$

and

$$g_{r\alpha} = \alpha \frac{\hat{F}\left(1+\beta_s^{-1}\right) + \hat{E}m}{1+\hat{E}(1-q)} \qquad g_{r0} = \alpha\beta_s(\langle r^2\rangle_G - \langle rv_0\rangle_G) = \alpha \tag{A.11}$$

$$g_{v0} = \alpha\beta_s(\langle rv_1\rangle_G - \langle v_1 v_\alpha\rangle_G). \tag{A.12}$$

Some of these factors only appear in certain combinations, however, although we have tried we have not made any significant simplification through further algebraic manipulation.

Appendix B. Calculation of our order parameters for the conventional demodulator

In this appendix, to lighten notation slightly we assume that the original message was $\boldsymbol{\sigma}$ while our estimator is $\hat{\boldsymbol{\sigma}}$. The conventional demodulator sets $\hat{\sigma}_i = \mathrm{sgn}(f_i)$ with

$$f_i = \alpha\sigma_i + \frac{1}{N}\sum_{t,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_t \eta_i^t \nu^t. \tag{B.1}$$

Since $\nu^t \sim \mathcal{N}(0, N/\beta_s)$ and we assume that $\left\{\eta_i^t, \sigma_i, \nu^t\right\}$ are all mutually independent random variables, we can use the central limit theorem to treat the second and third terms in (B.1). So the error probability for a single bit is the probability that a Gaussian $\mathcal{N}(0, 1)$ random variable is less than $-\sqrt{\alpha/\left(1+\beta_s^{-1}\right)}$ which gives

$$M_{CD} = \mathrm{Erf}\left[\sqrt{\frac{\alpha}{2\left(1+\beta_s^{-1}\right)}}\right]. \tag{B.2}$$

We can also see that $F_{CD} = \frac{1}{N}\sum_i f_i \mathrm{sgn}(f_i) = \mathbb{E}|f_i|$ is given by

$$F_{CD} = \int \mathrm{d}z \left|\alpha + \sqrt{\alpha\left(1+\beta_s^{-1}\right)}z\right|. \tag{B.3}$$

Now, we can write E_{CD} as

$$E_{CD} = \frac{1}{2}\sum_t \left(m_t^{CD}\right)^2 \qquad m_t^{CD} = \frac{1}{N}\sum_i \eta_i^t\hat{\sigma}_i. \tag{B.4}$$

Now,

$$\eta_i^s \hat{\sigma}_i = \eta_i^s \operatorname{sgn}\left[\alpha\sigma_i + \frac{1}{N}\sum_{t,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_t \eta_i^t \nu^t\right] \tag{B.5}$$

$$= \eta_i^t \operatorname{sgn}\left[\alpha\sigma_i + \frac{1}{N}\sum_{t\neq s,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_{t\neq s}\eta_i^t\nu^t\right] + \frac{1}{N}\sum_i\left(\sum_j \eta_j^s\sigma_j + \nu^s\right)$$
$$\times\, 2\delta\left(\alpha\sigma_i + \frac{1}{N}\sum_{t\neq s,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_{t\neq s}\eta_i^t\nu^t\right) + \mathcal{O}(N^{-1}) \tag{B.6}$$

where we have thrown away irrelevant terms and expanded the sgn function in a Taylor expansion (it is possible, although longer to check we obtain the same result without using this trick). We define

$$\gamma^t = \frac{1}{N}\sum_i \eta_i^t \operatorname{sgn}\left[\alpha\sigma_i + \frac{1}{N}\sum_{t\neq s,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_{t\neq s}\eta_i^t\nu^t\right] \tag{B.7}$$

$$\chi = \frac{1}{N}\sum_i 2\delta\left(\alpha\sigma_i + \frac{1}{N}\sum_{t\neq s,j\neq i}\eta_i^t\eta_j^t\sigma_j + \frac{1}{N}\sum_{t\neq s}\eta_i^t\nu^t\right) \tag{B.8}$$

$$= \frac{1}{2}\sum_\sigma \int \mathrm{D}z\, 2\delta\left[\alpha\sigma + \sqrt{\alpha\left(1+\beta_s^{-1}\right)}z\right] = \sqrt{\frac{2}{\pi\alpha\left(1+\beta_s^{-1}\right)}}\exp\left(-\frac{\alpha}{2\left(1+\beta_s^{-1}\right)}\right) \tag{B.9}$$

$$\kappa^t = \frac{1}{N}\sum_j \eta_j^t\sigma_j. \tag{B.10}$$

So up to irrelevant constants,

$$m_t^{CD} = \gamma^t + \kappa^t\chi + \sqrt{N\beta_s^{-1}}\chi z_1 \tag{B.11}$$

where above and in the following z_1, z_2, z_3 are $\mathcal{N}(0,1)$ random variables. The slight complication is that γ^t and κ^t are correlated since the latter is a trace over $\eta_i^t\sigma_i$ while the former is a trace over essentially $\eta_i^t\hat{\sigma}_i$—thus M_{CD} determines the degree of correlation and we find

$$E_{CD} = \frac{\alpha}{2}\left[1 + 2M_{CD}\chi + \chi^2\left(1+\beta_s^{-2}\right)\right]. \tag{B.12}$$

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